Fourier Analysis 04-06

Review:

Prop 1. Let $f \in M(\mathbb{R})$. Then the following hold:

① $f(x+h) \xmapsto{\mathcal{F}} \hat{f}(\xi) \cdot e^{2\pi i \xi h}$, $\forall h \in \mathbb{R}$.

② $f(x) \cdot e^{-2\pi i h x} \xmapsto{\mathcal{F}} \hat{f}(\xi + h)$, $\forall h \in \mathbb{R}$.

③ Let $\delta > 0$. Then

$$f(\delta x) \xmapsto{\mathcal{F}} \frac{1}{\delta} \hat{f}\left(\frac{\xi}{\delta}\right).$$

④ $f'(x) \xmapsto{\mathcal{F}} \hat{f}(\xi) \cdot (2\pi i \xi)$, $f' \in M(\mathbb{R})$.

④' $f^{(n)}(x) \xmapsto{\mathcal{F}} \hat{f}(\xi) \cdot (2\pi i \xi)^n$, if $f^{(j)} \in M(\mathbb{R})$, $1 \le j \le n$.

⑤ $-2\pi i x f(x) \xmapsto{\mathcal{F}} \dfrac{d \hat{f}(\xi)}{d \xi}$, $x f(x) \in M(\mathbb{R})$.

⑤' $(-2\pi i x)^n f(x) \xmapsto{\mathcal{F}} \dfrac{d^n \hat{f}(\xi)}{d \xi^n}$, $x^n f(x) \in M(\mathbb{R})$.

Example: $e^{-\pi x^2} \xmapsto{\mathcal{F}} e^{-\pi \xi^2}$.

§5.4 Inversion formula.

Thm 1. Let $f \in M(\mathbb{R})$. Suppose that $\hat{f} \in M(\mathbb{R})$.

Then

$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi)\, e^{2\pi i \xi x}\, d\xi$$

To prove the theorem, we introduce

Def. Let $f, g \in M(\mathbb{R})$. Set

$$f * g(x) = \int_{\mathbb{R}} f(x-y)\, g(y)\, dy.$$

Prop 2: Let $f, g \in M(\mathbb{R})$. Then

① $f * g = g * f$

② $f * g \in M(\mathbb{R})$

③ $\widehat{f * g}(\xi) = \hat{f}(\xi) \cdot \hat{g}(\xi)$.

pf of ②:

$$f * g(x) = \int_{\mathbb{R}} f(x-y)\, g(y)\, dy$$

$$= \int_{|y| \leq \frac{|x|}{2}} + \int_{|y| \geq \frac{|x|}{2}} f(x-y)\, g(y)\, dy$$

$$= (\mathrm{I}) + (\mathrm{II})$$

$$|(\mathrm{I})| \leq \int_{|y| \leq \frac{|x|}{2}} |f(x-y)|\, |g(y)|\, dy$$

(notice that $|x-y| \geq \frac{|x|}{2}$)

$$\leq \int_{|y| \leq \frac{|x|}{2}} \frac{A}{1+\left(\frac{|x|}{2}\right)^2} \cdot |g(y)|\, dy$$

$$\leq \frac{A}{1+\frac{|x|^2}{4}} \int_{\mathbb{R}} |g(y)|\, dy$$

$$\leq \frac{\tilde{A}}{1+|x|^2}$$

$$|(\mathrm{II})| \leq \int_{|y| \geq \frac{|x|}{2}} |f(x-y)| \cdot |g(y)|\, dy$$

$$\leq \int_{|y| \geq \frac{|x|}{2}} |f(x-y)| \cdot \frac{A}{1+\left(\frac{|x|}{2}\right)^2}\, dy$$

$$\leq \frac{A}{1+\left(\frac{|x|}{2}\right)^2} \cdot \int_{\mathbb{R}} |f(x-y)| \, dy$$

$$\leq \frac{A}{1+\left(\frac{|x|}{2}\right)^2} \int_{\mathbb{R}} |f(y)| \, dy$$

$$\leq \frac{\tilde{A}}{1+|x|^2}.$$

Hence $\displaystyle |f * g(x)| \leq \frac{A'}{1+|x|^2}.$

Also it is easy to show that $f * g$ is cts. ▨

<u>Def.</u> A family of $(K_t)_{t\in(a,b)} \subset M(\mathbb{R})$ is called a good kernel on $\mathbb{R}$ as $t \to t_0$, if

① $\displaystyle \int_{\mathbb{R}} K_t(x) \, dx = 1 \quad \forall \ t \in (a,b).$

② $\displaystyle \int_{\mathbb{R}} |K_t(x)| \, dx \leq M$ for some Const $M > 0$, $\forall \ t \in (a,b)$

③ $\forall \ \delta > 0,$

$$\int_{|x|\geq\delta} |K_t(x)|\,dx \to 0 \quad \text{as } t \to t_0.$$

Thm (Convergence Thm):

If $(K_t)_{t\in(a,b)}$ is a good kernel on $\mathbb{R}$, as $t \to t_0$, and $f \in \mathcal{M}(\mathbb{R})$, then

$$f * K_t(x) \rightrightarrows f(x) \quad \text{as } t \to t_0.$$

Thm (Multiplicative formula):

Let $f, g \in \mathcal{M}(\mathbb{R})$. Then

$$\int_{\mathbb{R}} f(x)\,\hat{g}(x)\,dx = \int_{\mathbb{R}} \hat{f}(x)\,g(x)\,dx.$$

(Fubini Thm: Let $F(x,y) \in C(\mathbb{R}^2)$.

Suppose one of the 3 integrations are finite

① $\iint_{\mathbb{R}^2} |F(x,y)|\,dx\,dy$;

② $\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |F(x,y)| \, dy \right) dx$;

③ $\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |F(x,y)| \, dx \right) dy$;

Then

$$\iint_{\mathbb{R}^2} F(x,y) \, dx dy = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} F(x,y) \, dy \right) dx$$
$$= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} F(x,y) \, dx \right) dy .$$

Now we prove the multiplicative formula:

Notice that

$$\int_{\mathbb{R}} f(x) \hat{g}(x) \, dx$$

$$= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x) g(y) e^{-2\pi i x y} dy \right) dx$$

$$\overset{\text{by Fubini}}{=} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x) g(y) e^{-2\pi i x y} dx \right) dy$$

$\left(\text{checking that } \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x)| |g(y)| \, dy \right) dx < \infty \right)$

$$= \int_{\mathbb{R}} \hat{f}(y)\, g(y)\, dy$$

$\boxed{/\!/\!/}$

Recall (Inversion Formula)

: If $f \in M(\mathbb{R})$ and $\hat{f} \in M(\mathbb{R})$, then

$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi)\, e^{2\pi i \xi x}\, d\xi .$$

Proof: We first prove that

$$f(0) = \int_{\mathbb{R}} \hat{f}(\xi)\, d\xi .$$

Set for $\delta > 0$,

$$K_\delta(x) = e^{-\pi \delta x^2} .$$

Notice that $$\widehat{K_\delta}(\xi) = \frac{1}{\sqrt{\delta}}\, e^{-\pi \xi^2/\delta} .$$

Fact: $\left(\widehat{K_\delta}\right)_{\delta > 0}$ is a good kernel as $\delta \to 0$.

- $$\int_{\mathbb{R}} \widehat{K_\delta}(x)\,dx = \int_{\mathbb{R}} \frac{1}{\sqrt{\delta}} e^{-\pi x^2/\delta}\,dx$$

$$\overset{\text{Letting } y=\frac{x}{\sqrt{\delta}}}{=\!=} \int_{\mathbb{R}} e^{-\pi x^2}\,dx$$

$$= 1$$

- $$\int_{\mathbb{R}} |\widehat{K_\delta}(x)|\,dx = 1$$

- $\forall\ \gamma > 0$

$$\int_{|x|\geq\gamma} |\widehat{K_\delta}(x)|\,dx = \int_{|x|\geq\gamma} \frac{1}{\sqrt{\delta}} e^{-\pi \frac{x^2}{\delta}}\,dx$$

$$\overset{\text{Letting } y=\frac{x}{\sqrt{\delta}}}{=\!=} \int_{|y|\geq\frac{\gamma}{\sqrt{\delta}}} e^{-\pi y^2}\,dy$$

$$\to 0 \quad \text{as} \quad \delta \to 0$$

Now by the convergence Thm,

$$f(0) = \lim_{\delta \to 0} f * \widehat{K_\delta}(0)$$

$$= \lim_{\delta \to 0} \int_{\mathbb{R}} f(x)\, \widehat{K_\delta}(-x)\, dx$$

$$= \lim_{\delta \to 0} \int_{\mathbb{R}} f(x)\, \widehat{K_\delta}(x)\, dx$$

$$\overset{\text{by Multiplicative formula}}{=\!=} \lim_{\delta \to 0} \int_{\mathbb{R}} \hat{f}(x)\, K_\delta(x)\, dx$$

$$\overset{\text{Dominated convergence Thm}}{=\!=} \int_{\mathbb{R}} \lim_{\delta \to 0} \hat{f}(x) K_\delta(x)\, dx$$

$$\left(K_\delta(x) = e^{-\pi \delta x^2} \right)$$

$$= \int_{\mathbb{R}} \hat{f}(x)\, dx$$

This proves the inversion formula at $x = 0$.

Now for any $x_0 \in \mathbb{R}$, define

$$f_{x_0}(x) = f(x + x_0).$$

Then

$$f_{x_0}(0) = \int_{\mathbb{R}} \widehat{f_{x_0}}(\xi)\, d\xi$$

$$\text{LHS} = f(x_0)$$

$$\text{RHS} = \int_{\mathbb{R}} \hat{f}(\xi)\, e^{2\pi i \xi x_0}\, d\xi$$

So we obtain

$$f(x_0) = \int_{\mathbb{R}} \hat{f}(\xi)\, e^{2\pi i \xi x_0}\, d\xi$$

$\blacksquare$.